On Extremal Compactifications Of Convergence Spaces

By
CHALLA JAGAN MOHAN RAO

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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CERTIFICATE

This is to certify that the work embodied in the thesis 'On Extremal Compactifications of Convergence Spaces' by Challa Jagan Mohan Rao has been carried out under our supervision and has not been submitted elsewhere for a degree or diploma

Feb 1975

[S. A Naimpally]
Professor,
Department of Mathematics,
Lakehead University,
Thunder Bay, Ontario, Canada

[R.S.L. Srivastava]
Professor,
Department of Mathematics
I.I T ,Kanpur,India

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C J M Rao

(C J M RAO)

CONTENTS

SYNOPSIS

Of 'On extremal compactifications of convergence spaces', a thesis submitted in partial fulfilment of the requirements for the Ph D degree by Challa Jagan Mohan Rao to the department of Mathematics, Indian Institute of Technology, Kanpur

Convergence is one of the important notions in Analysis and Topology. Beginning with the convergence of sequences, various convergences are studied in Mathematics. The abstract theory of convergence originated in the study of certain phenomena in Analysis and it was recognized that there are convergences which are non-topological. Axiomatic treatment of convergence is due to H P Fischer [4] who laid down the foundation for an independent study into such structures. D C Kent [7] studied a slight generalization of Limiterung of Fischer. This generalized structure known as convergence structure was shown subsequently by Kent to be more complete one from the point of view of the lattice of convergence structures.

Our area of interest is compactification problems of convergence spaces. Quite some work has already appeared on the compactification theory of these spaces. In this thesis our main concern is with the extremal compactifications of convergence spaces. We study extremal Hausdorff and regular compactifications, extremal problems of Richardson compactification and realcompactifications. Chapter One contains a brief summary of the known results that we need in the present thesis.

Our own work begins in Chapter Two, where we study Hausdorff compactifications We give a characterization of the class of convergence spaces having the smallest Hausdorff compactifications We first construct a one-point Hausdorff compactification that always exists, and define the concept of <u>local compactness</u> for convergence spaces We show that a Hausdorff convergence space has smallest Hausdorff compactification if and only if it is locally compact In case the convergence space is a locally compact topological space, then the one-point Hausdorff compactification so constructed becomes topological and hence coincides with the well known Alexandroff one-point topological compactification We next study the problem of the existence of the largest Hausdorff compactification. We show that if a convergence space has such a compactification, it must be locally compact Moreover, a necessary and sufficient condition for the existence of the largest Hausdorff compactification is that the convergence space has at most finitely many non-convergent ultrafilters

In Chapter Three we study Richardson compactification [14]- a Stone-Čech type of Hausdorff compactification of a convergence space This has the property that every continuous function from the convergence space into a regular (regular we mean including Hausdorff) compact convergence space has a unique continuous extension to this compactification But, in general, Richardson compactification is not necessarily regular (and so fails to be the largest regular compactification) and also, the regularity requirement of the range space (for extension of continuous functions) can not be relaxed in general (and so fails to be the largest

Hausdorff compactification) Under these circumstances, it is natural to ask when Richardson compactification is regular or the largest Hausdorff compactification These problems are studied in this chapter Our results supplement those of R.J. Gazik [5]

For convergence spaces, Hausdorff compactifications and regular compactifications form two regions of study and their behaviour also differs considerably from each other. G D Richardson and D C Kent [15] have characterized convergence spaces having regular compactifications, and have further shown that such spaces also possess the largest regular compactifications We continue this study with a characterization of convergence spaces having smallest regular compactifications We then define an equivalence relation on the class of all regular compactifications of a convergence space and show that each equivalence class has a smallest and a largest member There is a one to one correspondence between these equivalence classes and Hausdorff topological compactifications of the pre-topological modification of the convergence space (this modification yielding a Tychonoff topological space)

In Chapter Five we study m-ultracompactifications of convergence spaces m-ultracompactness for T_1 topological spaces have been defined by J van der Slot [16], for an infinite cardinal m $\aleph_0$-ultracompactness is equivalent to compactness and $\aleph_1$-ultracompactness is equivalent to realcompactness for countably compact normal spaces. We extend this to convergence spaces We construct a Hausdorff m-ultracompactification having the extension property of continuous functions to regular m-ultracompact space We find conditions under which it is largest

Hausdorff and largest regular ultracompactification respectively

We also find a condition under which a convergence space has smallest m-ultracompactification

Several open problems are mentioned through out the thesis

INTRODUCTION

In this Thesis we deal with convergence as a primitive notion, i e we study an abstract set in which convergence of filters is postulated Such a theory was originally introduced for the study of certain non-topological phenomena in Analysis In 1948 Choquet [1] in his study of relations between two non-uniformizable topological spaces, was led in a natural way to the concept of pseudo-topology on the set of closed subsets of a topological space Thereafter Sonner [17] gave a set of axioms for limit spaces, and established polarity between limit spaces and topological spaces Kowalsky [11] studied the completion problems for such spaces A comprehensive exposition of these concepts was given by Fischer [4], and this is the foundation of all subsequent studies in this direction Kent [7], [9] studied a slight generalization of the Limiterung of Fischer This generalization known as convergence structure was shown by Kent in his further studies to be more complete one from the point of view of the lattice of convergence structures Some of the examples of convergence structures, not derivable from topological structures are continuous convergence on a collection of continuous maps from one topological space to another (Cook and Fischer [3]), almost everywhere convergence of measurable functions (Taylor [18]), order convergence in a lattice (Kent [8])

Our area of interest is compactification problems of convergence spaces Quite some work has already appeared on compactification theory of these spaces Cochran [2] in his Doctoral Thesis constructed a Hausdorff compactification T of a convergence space S such that every bounded real valued continuous function from S has a unique continuous extension to T Wyler [19] showed that for each Hausdorff convergence space S, there exists a regular (regular we mean including Hausdorff) compact convergence space S*, and a continuous map j from S to S* such that every continuous map from S into a regular compact convergence space T has a unique continuous extension to S* But, in this case S is not necessarily embedded in S*, and hence S* is not a compactification of S in the usual sense Similar extensions were studied by Ramaley [12] , Ramaley and Wyler [13] Richardson [14] showed that every Hausdorff convergence space S can be densely embedded in a compact Hausdorff convergence space (X_1, i) such that if f is a continuous function from S into a regular compact convergence space Y, then there exists a unique continuous function h from X_1 to Y such that $h \circ i = f$ This compactification is called <u>Richardson compactification</u> In general, this compactification is not regular, hence it fails to be the largest regular compactification of S Moreover, the requirement that the range space be regular (for extension of continuous functions) can not be relaxed in general, and so this is not necessarily the largest Hausdorff compactification These

observations lead to the investigation into the problems involving extremal compactifications of convergence spaces, such as (i) which convergence spaces have largest (smallest) Hausdorff compactifications, (ii) when is Richardson compactification the largest Hausdorff (regular) compactification, (iii) which convergence spaces have (extremal) regular compactifications etc In the present work, we bring together solutions to these and other related problems

The above remarks suggest the possibility that Hausdorff compactifications and regular compactifications constitute two separate domains of investigation linked together by the Richardson compactification Our study together with those of Richardson and Kent [15] confirms this For this reason, we deal with these two types of compactifications in two different chapters and a chapter dealing Richardson compactification coming in between

In the first chapter we collect relevent notations, definitions, and the known results required in the Thesis The second chapter is on Hausdorff compactifications We begin with a characterization of the class of convergence spaces having the smallest Hausdorff compactifications This is done through the construction of a one-point Hausdorff compactification which always exists, and the concept of <u>local compactness</u> (a convergence space is locally compact if it is open in each of its Hausdorff compactifications) The above mentioned characterization is a convergence space has the

smallest Hausdorff compactification if and only if it is locally compact In case the convergence space is a locally compact topological space the above compactification becomes topological, and hence coincides with the well known Alexandroff one-point topological compactification We next obtain in this chapter a characterization of convergence spaces having the largest Hausdorff compactifications We show that such a convergence space is necessarily locally compact and hence has the smallest Hausdorff compactification In fact we prove that a convergence space has the largest Hausdorff compactification if and only if it has at most finitely many non-convergent ultrafilters, and hence all of its Hausdorff compactifications are finite point compactifications

Chapter Three concerns the Richardson compactification Gazik [5] has obtained a necessary and sufficient condition that the Richardson compactification of a regular convergence space is regular (and hence the largest regular compactification) A special case of Gazik's result is the Stone-Cech compactification of a Tychonoff topological space We continue Gazik's work and obtain a necessary and sufficient condition that the Richardson compactification of a convergence space is its largest Hausdorff compactification Naturally this involves local compactness and some condition on non-convergent ultrafilters

Chapter Four deals with regular compactifications Richardson and Kent [15] have obtained a characterization of convergence spaces having regular compactifications, and have shown that each such convergence space has a largest regular compactification We continue this work and obtain a characterization of those convergence spaces which have smallest regular compactifications Further we define an equivalence relation on the set of all regular compactifications of a convergence space, and show that each equivalence class has a smallest member and a largest member And there is a one to one correspondence between these equivalence classes and Hausdorff topological compactifications of the pre-topological modification of the convergence space (this modification yielding a Tychonoff topological space) This shows that regular compactifications of a convergence space are more closely related than Hausdorff compactification to the topological case

Chapter Five is on m-ultracompactifications m-ultracompactness for T_1 topological spaces was defined by J vander Slot [16], for an infinite cardinal m $\aleph_0$-ultracompactness is equivalent to compactness Real compactness and $\aleph_1$-ultracompactness for countably compact normal spaces are equivalent We extend this definition to convergence spaces and show that it works well with Richardson method of compactification In the first section we find condition so that a convergence space may have smallest

m-ultracompactification and in section two we construct an m-ultracompactification in a manner similar to Richardson compactification Then we find conditions so that this m-ultracompactification may become the largest Hausdorff m-ultracompactification and largest regular m-ultracompactification respectively Finally we mention some open problems

CHAPTER I

PRELIMINARIES

In this chapter we collect all notations, definitions and results that we need in the present thesis Most of the results carry reference numbers indicating the place where these have been proved Rest of the results whose existence could not be traced by us in the literature, are proved here Besides these, Richardson compactification [14] and results of Gazik [5] are described briefly This chapter is divided into three section, namely notations, definitions and results

1 *Notations*

If S is a set and x belongs to S, then $\underline{x}$ will denote the ultrafilter generated by x, namely $\{A \subset S \mid x \in A\}$ Filters on S are usually denoted by $\mathcal{F}$, $\mathcal{G}$ etc The set of all filters by $F(S)$ and set of all subsets by $P(S)$ $\underline{\mathcal{F} \wedge \mathcal{G}}$ will denote the filter whose base is $\{F \cup G \mid F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$ If f is a function from S to S', then $\underline{f(\mathcal{F})}$ will denote the filter on S' whose base is $\{f(F) \mid F \in \mathcal{F}\}$ If S is a subset of some set T, then filters on T are given by ϕ, ψ etc If every member of ϕ intersects S, that is, trace of ϕ on S exists, then this trace is denoted by $\underline{\phi \cap S}$ The filter $\phi \cap S$ on S will generate a filter on T which is denoted

by $[\phi \cap S]_T$, when there is no possibility of doubt, the subscript T is dropped We would like to point out that $[\phi \cap S]_T \geq \phi$ Similarly if $S \subset T$ and F is a filter on S, then $[F]_T$ denotes the filter on T generated by F Again we note that $[F]_T \cap S = F$

2 *Definitions*

A convergence structure q on a set S is a function from S into $P(F(S))$ satisfying the following conditions

C1 $\dot{x}$ belongs to $q(x)$ for each x in S

C2 F belongs to $q(x)$ implies $F \wedge \dot{x}$ belongs to $q(x)$ for each x in S

C3 F belongs to $q(x)$ and $G \geq F$ implies G belongs to $q(x)$ for each x in S

If F belongs to $q(x)$, we say F q-converges to x When the convergence structure on S is known, we write F S-converges to x, or simply F converges to x when there is no possibility of doubt A subset A of S is said to be open if each filter F on S which converges to a point x in A, contains A A is said to be closed if complement of A in S is open The set of all closed subsets of S is denoted by C_S For a filter F on S $F \cap C_S$ denotes the set of all closed subsets of S in F A subset B of S is said to be the closure of a subset A of S if B is the set of all x in S such that there exists a filter F on S containing A converging to x The closure of A in S is denoted by $C\ell_S A$ For a given filter F on S, $C\ell_S F$ is the filter which is generated by $\{C\ell_S F \mid F \in F\}$ For x in S we denote by $V_S(x)$

the intersection filter of all filters that converge to x. A point x in S is said to be *adherence point* of a filter F on S if there exists an ultrafilter $G \geq F$ such that G converges to x. A convergence structure is said to be *Hausdorff* if each filter on S converges at most to one point. A Hausdorff convergence space is said to be *regular* if F converges to x implies $C\ell_S F$ converges to x. If F is an ultrafilter on S, then F is said to be *m-ultrafilter* for an infinite cardinal m, if $F \cap C_S$ has m-intersection property (i.e. each subcollection of closed subsets of F of cardinal $< m$ has non-empty intersection). S is said to be *m-ultracompact* if each m-ultrafilter on S converges. $\aleph_0$-ultracompact space is said to be *compact*. Therefore, S is compact if and only if every ultrafilter on S converges. A function f from a convergence space S to another S' is said to be *continuous* if a filter F on S is S-convergent to x in S, then $f(F)$ is S'-convergent to $f(x)$. A one-to-one bicontinuous function from S to S' is said to be *isomorphism* and S and $f(S)$ are said to be *isomorphic*. A subset S of a convergence space T is said to be *dense* in T if for each x in T, there exists a filter on T converging to x and containing S. If S and S' denote two convergence spaces on the same set, we say *S is finer than S'* (or *S' is coarser than S*) if for each filter F on S, F S-converges to x implies F S'-converges to x, and it is denoted by $S \geq S'$. Under this ordering it is easily seen that the set of all convergence structures of a set is a complete lattice, where if $C(S)$ is a set of such structures, then *$\inf C(S)$* and *$\sup C(S)$* are given as follows

a filter $\mathcal{F}$-inf $C(S)$ - convergesto x if there exists a convergence structure q in $C(S)$ such that $\mathcal{F}$ q-converges to x,

$\mathcal{F}$-sup $C(S)$ - converges to x if $\mathcal{F}$ q-converges to x for each q in $C(S)$

If S is a Hausdorff convergence space and T a compact Hausdorff convergence space and there exists a bicontinuous one to one function from S into T such that f(S) is dense in T, then T is said to be a Hausdorff <u>compactification</u> of S, and is denoted by <u>(T,f)</u> If (T,f) and (T',g) are two Hausdorff compactifications of S, then we say (T,f) is <u>finer</u> than (T',g) and denote <u>$(T,f) \geq (T',g)$</u> if there exists a continuous function h from T onto T' such that $h \circ f = g$ We say (T,f) and (T',g) are <u>equivalent</u> compactifications of S if $(T,f) \geq (T',g)$ and $(T',g) \geq (T,f)$ Hausdorff compactification (T,f) of S is said to be <u>regular</u> if T is a regular convergence space If S is a Hausdorff convergence space and T an m-ultracompact Hausdorff space, and f is a bicontinuous one to one function from S to T such that f(S) is dense in T, we say (T,f) is an <u>m-ultracompactification</u> of S If (T,f) and (T',g) are two m- ultracompactifications of S, then we write <u>$(T,f) \geq (T',g)$</u> if there exists a continuous function h from T into T' such that $h \circ f = g$, and say (T,f) is <u>finer</u> than (T',g) If S is a convergence space such that $V_S(x)$ converges to x for each x, we say S is a <u>pretopological</u> space The finest pretopological space on the set S coarser than S is denoted by <u>πS</u> This is defined as $\mathcal{F}$, a filter on S will πS-converge to x in S if and only if $\mathcal{F} \geq V_S(x)$ This is called <u>pretopological</u> <u>modification</u> of S Similarly the finest topology on the set S coarser than S is called

the topological modification of S and is denoted by λS λS is defined as a filter F on S will λS-converge to x in S if and only if $F \geq B_S(x)$, where $B_S(x)$ is the filter generated by the sets U in $V_S(x)$, which have the property that y belongs to U implies U belongs to $V_S(y)$ If S is a subset of a convergence space T then subspace convergence structure on S is defined as follows a filter F on S will S-converge to x in S if and only if $[F]_T$ T-converges to x

3 *Results*

We begin with two simple results of a technical nature We will make use of these often without explicit mention

Theorem 1 1 If F_1 and F_2 are two filters on a set S and G is an ultrafilter on S such that $G \geq F_1 \wedge F_2$, then $G \geq F_1$ or $G \geq F_2$

Proof Let F_1 be in F_1 and F_2 in F_2, such that both F_1 and F_2 does not belong to G Now $F_1 \cup F_2$ belongs to $F_1 \wedge F_2$ and therefore to G But G is an ultrafilter, hence either F_1 or F_2 belongs to G Hence we get a contradiction

Theorem 1 2 A set S is finite if and only if every ultrafilter on S is a point ultrafilter

Proof Let S be a finite set and F be an ultrafilter, then S belongs to F, hence some $\{x\}$ for x in S belongs to S, that is, F is a point ultrafilter On the other hand if S is infinite then S-complements of finite sets in S will constitute a filter base, let F be an

ultrafilter finer than this, then F is not a point ultrafilter

Theorem 1 3 [Fischer [4], Theorem 10, p 278] If S is a pre-topological compact convergence space and S* is a Hausdorff convergence space such that $S^* \leq S$, then $S^* = S$

Theorem 1 4 [Fischer [4], Theorem 13, p 279] If S is a convergence space and x belongs to $C\ell_S$ A for $A \subset S$, then $U \cap A \neq \emptyset$, for every U in $V_S(x)$

Theorem 1 5 [Kent [6], Lemma 1, p 3] If S is a convergence space, then $A \subset S$ is S-closed if and only if A is λS-closed if and only if $C\ell_S A = A$

Theorem 1 6 [Fischer [4], Theorem 15, p 279] If S is a Hausdorff convergence space, then for each x in S, $\{x\}$ is closed

Theorem 1 7 [Kent [6], Theorem 2, p 5] If S is a compact Hausdorff space and A is a closed subset of S, then A is compact

Theorem 1 8 [Kent [6], Theorem 2, p 5] If S is a Hausdorff convergence space and A is a compact subset of S, then A is closed in S

Theorem 1 9 [Kent [6], Theorem 1, p 16] Let x be in $A \subset S$, then $V_S(x) \cap A = V_A(x)$

Theorem 1 10 [Kent and Richardson [10], Theorem 1 1 p 488] A convergence space S is minimal Hausdorff if and only if S is a compact Hausdorff convergence space such that whenever every ultrafilter finer than F converges to some x in S, then so does F

<u>Theorem</u> 1 11 [Fischer [4], Theorem 1, p 283] Composition of two continuous functions is continuous

<u>Theorem</u> 1 12 [Fischer [4], Theorem 2, p 283] Restriction of a continuous function to a subspace is continuous

<u>Theorem</u> 1 13 [Kent [6], Theorem 4, p 9] Continuous image of a compact space is compact

<u>Theorem</u> 1 14 [Fischer [4], Theorem 5, p 284] If f is a continuous function from a convergence space S to another S*, then $f(\mathcal{C}\ell_S A) \subset \mathcal{C}\ell_{S*} f(A)$, for each A S

<u>Theorem</u> 1 15 [Kent [6], Theorem 2, p 8] If f is a continuous function from S to S*, then A ⊂ S* is S*-closed implies $f^{-1}(A)$ is S-closed

<u>Theorem</u> 1 16 [Kent [9], Corollary to Theorem 1, p 199] If f is a continuous function from S to S*, then so is f from πS to πS*

<u>Theorem</u> 1 17 [Kent [7], Theorem 4, p 130] If S is a convergence space, then S-closure operator is idempotent if and only if πS is a topology

<u>Theorem</u> 1 18 [Kent and Richardson [10], Theorem 1 9, p 490] If f is a continuous bijection from a minimal Hausdorff space into a Hausdorff space S*, then f is an isomorphism

<u>Theorem</u> 1 19 [Fischer [4], Theorem 6, p 284] If f and g are two continuous functions from S to S* such that f and g agree on a dense subset A of S and S* is Hausdorff, then f = g

From this theorem we derive the following two results which we will use often

Theorem 1 20 If S is a dense subspace of a converges space T Then a continuous function from S into a Hausdorff convergence space S* has at the most one extension to T, that is, whenever such an extension exists, it is unique

Theorem 1 21 Let (T_1,f_1) and (T_2,f_2) be two Hausdorff compactifications (or m-ultracompactifications) of a convergence space S such that each is finer than the other, then the two are equivalent, that is, there exists an isomorphism between T_1 and T_2 such that $h \circ f_1 = f_2$

Theorem 1 22 Let S be a convergence space and T_1, T_2 be Hausdorff convergence spaces and f_1 and f_2 be isomorphisms from S to T_1 and T_2 such that $f_1(S)$ and $f_2(S)$ are dense in T_1 and T_2 respectively Let h be a continuous function from T_1 to T_2 such that $h \circ f_1 = f_2$ Then $h(f_1(S)) = f_2(S)$ and $h(T_1-f(S)) \subset T_1-f_2(S)$, and when h is an onto function then $h(T_1-f(S)) = T_2 - f_2(S)$

Proof The first property follows from the fact $h \circ f_1 = f_2$ To prove the second we need only show that $h(T_1-f_1(S)) \cap h(f_1(S)) = \emptyset$ Let $S_1 = h^{-1} \circ h(f_1(S))$, it is sufficient to show that $S_1 = f_1(S)$ Now $f_1(S) \subset S_1 \subset T_1$, S_1 is Hausdorff and $f_1(S)$ is dense in S_1 Let g be the inverse of the map h restricted to $f_1(S)$ and g_1 be $g \circ (h|S_1)$, then g_1 restricted to $f_1(S)$ is identity and hence g_1 is identity from S_1 to S_1 Since $g(S_1) \subset f_1(S)$ we have $S_1 = f_1(S)$

$$\begin{tikzcd} T_1 \arrow[rr, dashed, "h"] & & T_2 \\ & S \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

This result we will use in the context of compactifications (or m-ultracompactifications in general)

Theorem 1 23 [Richardson and Kent [15], Proposition 1, p 572] The closure operator of a compact regular convergence space is idempotent

<u>Theorem</u> 1 24 [Richardson and Kent [15], Proposition 3, p 572] If A is subspace of a compact regular convergence space, then πA is Hausdorff and topological

Now we describe Richardson compactification [14], of a Hausdorff convergence space S Let X_1 be the set of all x for x in S and all nonconvergent ultrafilters on S Let i be the mapping from S to X_1 defined as $i(x) = x$ Let $\mathcal{F}$ be a filter on S, then $\hat{\mathcal{F}}$ is the filter on X_1 whose base is $\{\hat{F} \mid F \in \mathcal{F}\}$, where $\hat{F} = \{\mathcal{H} \in X_1 \mid F \in \mathcal{H}\}$ A convergence structure is defined on the set X_1 (and the resulting space is denoted by X_1 as follows

let ϕ be a filter on X_1, then

ϕ X_1-converges to x if and only if $\phi \geq \hat{\mathcal{F}}$ for some filter $\mathcal{F}$ on S which S-converges to x

ϕ X_1-converges to v if and only if $\phi \geq \hat{v}$, where v is a nonconvergent ultrafilter on S

The space (X_1, i) is the Richardson compactification

In case X_1 consists of all x for x in S and all nonconvergent

m-ultrafilters on S, then with a convergence structure defined same as above is an m-ultracompact space (Chapter Five of this thesis)

We will now give proof of the following two results which have been proved by Gazik [5] The proofs are valid for both Richardson compactification and m-ultracompactification described above, though Gazik dealt only the compactification case

Let us denote by ρS the Richardson compactification (or m-ultracompactification described above) of S Let ϕ be a filter on ρS If A belongs to ϕ and f is any function from A to P(S) for which $f(D) \in D$ for each $D \in A$, we define $A(f) = \cup\{f(D) \mid D \in A\}$ Since ϕ is a filter on ρS, $A(f)$ for $A \in \phi$, constitute a filter base on S, let us denote it by $K(\phi)$ the filter generated by this base

<u>Theorem</u> 1 25 [Gazik [5], Lemma 1, p 2] Let ϕ be a filter on ρS, $\alpha \in \rho S$ and ϕ ρS-converges to α Then $K(\phi) = \alpha$, if α is nonconvergent and $K(\phi)$ S-converges to x if $\alpha = x$

<u>Proof</u> Let α be nonconvergent If $A \in \alpha$, by definition ρS, there exists an $H \in \phi$ such that $H \subset \hat{A}$ If f is defined on H as $f(h) = A$ for each h in H, then $H(f) = \cup\{f(h) \mid h \in H\} = A \in K(\phi)$ Hence $K(\phi) \geq \alpha$ But α is ultrafilter, hence $\alpha = K(\phi)$ If $\alpha = x$, then $\phi \geq \hat{F}$ for some filter F on S which S-converges to x Hence similar to above $K(\phi) \geq \alpha$

<u>Theorem</u> 1 26 [Gazik [5], Lemma 2, p 2] If $A \subset S$, then $c\ell_{\rho S}(\hat{A}) \subset (c\ell_S(A))^{\wedge}$

<u>Proof</u> Let $\alpha \in C\ell_{\rho S}(\hat{A})$ and α is nonconvergent Then there exists a filter ϕ on ρS such that $\hat{A} \in \phi$ and ϕ ρS-converges to α Hence by Theorem 1 25, $K(\phi) = \alpha$, so $\hat{A}(f) \in K(\phi) = \alpha$, where f is defined on $\hat{A}$ by $f(h) = A$ for $h \in \hat{A}$ Thus $A = \hat{A}(f) \in \alpha$ It follows that $\alpha \in \hat{A}$, so $\alpha \in (C\ell_S(A))^{\wedge}$ Let $\alpha = x \in C\ell_{\rho S}(\hat{A})$ Then there is a filter ϕ on ρS such that ϕ ρS-converges to x and $\hat{A} \in \phi$ By Theorem 1 25, $K(\phi)$ S-converges to x With f defined on $\hat{A}$ by $f(h) = A$ for $h \in \hat{A}$, we have $A = \hat{A}(f) \in K(\phi)$ Hence $x \in C\ell_S(A)$, therefore, $x \in (C\ell_S(A))^{\wedge}$

CHAPTER II

On HAUSDORFF COMPACTIFICATIONS

Introduction

Since every Hausdorff convergence space has a Hausdorff compactification, characterization of the class of Hausdorff convergence spaces having smallest Hausdorff compactifications, or likewise having largest Hausdorff compactifications, is of some interest In this chapter we investigate such problems Firstly, in section one we obtain a one point Hausdorff compactification of a convergence space This compactification always exists without needing any additional conditions on the underlying convergence space Using this one point compactification, we obtain a characterization of the class of convergence spaces which possess the smallest Hausdorff compactifications This characterization is in terms of local compactness, a notion that has been defined in this section This definition of local compactness which has been defined in analogy with topological case, is an 'external' one, in the sense that we demand the space to be an open subspace of each of its Hausdorff compactifications We do not know any 'internal' characterization, it has been proposed here as an open problem We also show that the one point compactification constructed in this section is 'proper', in the sense that if we take the underlying convergence space to be a locally compact Hausdorff topological space, then this

compactification becomes topological, and hence coincides with the Alexandroff one point compactification In section two we obtain a characterization of the class of Hausdorff convergence spaces which have the largest Hausdorff compactifications For this we use the Richardson compactification [14] , construction of which has been outlined in Chapter One Firstly, we show that any convergence space having the largest Hausdorff compactification is locally compact in the sense defined in this chapter And hence, each such convergence space also possesses the smallest Hausdorff compactification In fact we show that a convergence space has the largest Hausdorff compactification if and only if it has atmost finitely many non-convergent ultrafilters, as a consequence all of its Hausdorff compactifications are finite point compactifications

1 *Smallest Hausdorff Compactification*

In this section our aim is to obtain a characterization of the class of Hausdorff convergence spaces which possess the smallest Hausdorff compactifications

<u>The one point Hausdorff compactification</u>

Let S be a Hausdorff non-compact convergence space Let $T = S \cup \{z\}$, where z does not belong to S We make T a convergence space by

defining a convergence structure as follows

let ϕ be a filter on S, then

ϕ T-converges to z if and only if $\phi \geq \psi \wedge z$, where ψ is any filter on T such that $\psi \cap S$ has no S-adherence points

ϕ T-converges to x in S if and only if S belongs to ϕ and $\phi \cap S$ S-converges to x

Let i be the inclusion map from S into T We will show that (T, i) is a Hausdorff compactification of S The fact that T is a convergence space is simple to verify Let ϕ be an ultrafilter on T which is T-converging to some x in S and also T-converging to z Then, $\phi \geq \psi \wedge z$, where ψ is a filter on T such that $\psi \cap S$ has no S-adherence points Since ϕ T-converges to x in S, $\phi \neq z$, therefore $\phi \geq \psi$ But this is a contradiction Hence, T is a Hausdorff convergence space In order to see that T is compact, let ϕ be an ultrafilter on T If $\phi = z$, we are through If $\phi \neq z$, then S belongs to ϕ Now, $\phi \cap S$ either S-converges to some x in S, or is a non-convergent ultrafilter on S In the first case ϕ T-converges to x and in the second case ϕ T-converges to z i is obviously a bicontinuous one to one function from S into T Since S is a non-compact convergence space, it has a non-convergent ultrafilter F If we denote by ψ, the filter generated by $i(F)$ on T, then by definition, ψ T-converges to z, and hence $i(S)$ is dense in T

Local compactness

We give a definition of local compactness for a Hausdorff convergence space which is analogous to the topological case We would like to recall that a Tychonoff topological space is locally compact if and only if it is open, as a subspace, in each of its Hausdorff topological compactifications

Definition 2 1 A Hausdorff convergence space is locally compact if and only if it is open, as subspace, in each of its Hausdorff compactifications

Open problem Find an internal characterization of local compactness for convergence spaces

Characterization

Now we are in a position to give a characterization of the class of Hausdorff convergence spaces having the smallest Hausdorff compactifications

Theorem 2 2 A Hausdorff non-compact convergence space has the smallest Hausdorff compactification if and only if it is locally compact

<u>Proof</u> Let S be a Hausdorff non-compact locally compact convergence space, and (T, i) be the one point Hausdorff compactification of S constructed above We will show that in this case (T, i) is the smallest Hausdorff compactification of S Let (T', f) be an arbitrary Hausdorff compactification of S Let us define a function h from T' onto T as follows

$$h(y) = i \circ f^{-1}(y) \quad , \quad \text{for } y \text{ in } f(S)$$

$$\quad = z \qquad\qquad , \quad \text{for } y \text{ in } T' - f(S)$$

Therefore h is a function such that the following diagram commutes

$$\begin{tikzcd} T' \arrow[rr, "h"] & & T \\ & S \arrow[ul, "f"] \arrow[ur, "i"'] & \end{tikzcd}$$

We will show that h is a continuous function from T' onto T The fact that the function is well defined is obvious Let ϕ be a filter on T' which is T'-converging to y in $f(S)$ Now, S being locally compact, $f(S)$ belongs to ϕ Hence, $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$ Therefore, $i \circ f^{-1}(\phi \cap f(S))$ T-converges to $i \circ f^{-1}(y) = h(y)$ Since $h(\phi) = i \circ f^{-1}(\phi \cap f(S))$, we are through Next, let ϕ T'-converge to y in $T'-f(S)$ We have the following two possibilities (a) the trace of ϕ on $f(S)$ does not exist, then $T'-f(S)$ belongs to ϕ, hence $h(\phi) = z$ which T-converges to $z = h(y)$, (b) the trace of ϕ on $f(S)$ exists, then $\phi \cap f(S)$ has no $f(S)$-adherence points because ϕ T'-converges to y in $T'-f(S)$

and T' is Hausdorff Therefore, $f^{-1}(\phi \cap f(S))$ has no S-adherence points, and in turn, $i \circ f^{-1}(\phi \cap f(S))$ has no $i(S)$-adherence points But $i \circ f^{-1}(\phi \cap f(S)) = h(\phi \cap f(S)) = h(\phi) \cap i(S)$, hence $h(\phi)$ T-converges to $z = h(y)$ Since $h \circ f = i$, and the uniqueness of h follows from Theorem 1 20, we have $(T',i) \geq (T,i)$

Next, let S be a non-compact convergence space having the smallest Hausdorff compactification Let (T_1,g_1) be the smallest Hausdorff compactification of S Since, the one point Hausdorff compactification (T,i) always exists, we have $(T,i) \geq (T_1,g_1)$ Now, by Theorem 1 22, we conclude that (T_1,g_1) is a one point compactification of S Hence, S is open in its smallest Hausdorff compactification We will now show that S is open in each of its Hausdorff compactifications This we do as follows let (T_2,g_2) be an arbitrary Hausdorff compactification of S Since $(T_2,g_2) \geq (T_1,g_1)$, there exists a continuous function h from T_2 onto T_1 such that $h \circ g_2 = g_1$ Now, by Theorem 1 22, $h^{-1}(T_1-g_1(S)) = T_2-g_2(S)$ And $T_1-g_1(S)$ is T_1-closed being a singlton by Theorem 1 6, hence by Theorem 1 15, $T_2-g_2(S)$ is T_2-closed, and thus $g_2(S)$ is open implying thereby that S is locally compact This completes the proof of the theorem

Remarks on the one point compactification

We note the following two facts, which we obtain by observing the proof of the above theorem

(a) The one point Hausdorff compactification (T,i) of a convergence space S is coarser than any Hausdorff compactification of S in which S is open. This follows from the proof of 'if' part of the above theorem. This implies, in particular, that (T,i) is coarser than any finite point compactification of S. On the other hand if some Hausdorff compactification (X,f) of S is such that $(X,f) \geq (T,i)$, then using Theorem 1.22, one can show that S is open in X.

<u>Theorem</u> 2.3. A Hausdorff compactification (X,f) of S is finer than the one point Hausdorff compactification (T,i) of S if and only if S is open in X.

(b) If some Hausdorff compactification of S is coarser than the one point Hausdorff compactification (T,i) of S, then it follows from the proof of the 'only if' part of the Theorem 2.2, that this compactification must be a one point compactification. But in such a case, as it has been pointed out in (a) above, it must be finer than (T,i), and hence by Theorem 1.21, it is equivalent to (T,i).

<u>Theorem</u> 2.4. If some Hausdorff convergence space has the smallest Hausdorff compactification, then it must be (T,i).

Topological case

Let S be a locally compact (topological sense) Hausdorff topological space Let us denote by (T',j) the Alexandroff one point Hausdorff topological compactification of S Let (T,i) be the one point Hausdorff compactification (convergence sense) of S as constructed earlier Then, by remark (a) above $(T',j) \geq (T,i)$ Now, T' is a compact Hausdorff topological space, hence by Theorem 1 10, T' is a minimal Hausdorff space in the convergence sense Let h be the continuous function from T' onto T such that $h \circ j = i$ Now, both T' and T being one point compactifications of S, by Theorem 1 22, h is a continuous bijection from T' onto T Therefore, by Theorem 1 18, h is an isomorphism and hence $(T',j) = (T,i)$

Theorem 2 5 If S is a locally compact Hausdorff topological space, then (T,i) is its Alexandroff one point Hausdorff topological compactification

Now combining Theorem 2 4 and Theorem 2 5, we have the following result

Theorem 2 6 If a locally compact Hausdorff topological space has a smallest Hausdorff compactification in the convergence sense, then this compactification is topological, and in fact is the Alexandroff one point compactification

2 *Largest Hausdorff Compactification*

In this section we obtain a characterization of the class of Hausdorff convergence spaces which possess the largest Hausdorff compactifications

Local compactness

Firstly, we prove that any Hausdorff convergence space having the largest Hausdorff compactification is necessarily locally compact, and hence also possesses the smallest Hausdorff compactification

Theorem 2 7 If a Hausdorff convergence space has the largest Hausdorff compactification, then it is locally compact

Proof Let S be a Hausdorff convergence space having the largest Hausdorff compactification (X,f) Let (T,i) denote the one point Hausdorff compactification of S constructed in section one Then by Theorem 2 3, we know that $f(S)$ is open in X Now, we will show that S is open in each of its Hausdorff compactifications Let (Y,g) be an arbitrary Hausdorff compactification of S Then $(X,f) \geq (Y,g)$ From Theorem 1 22, we know that if h is the continuous function from X onto Y such that $h \circ f = g$, then $h(X-f(S)) = Y-g(S)$ Now, $f(S)$ being open, $X-f(S)$ is closed, and by Theorem 1 7, it is compact Hence $Y-g(S)$ being a continuous image of a compact set is

compact by Theorem 1 13 Since Y is Hausdorff, by Theorem 1 8, Y-g(S) is closed, and thus g(S) is open in Y This proves S is locally compact

Characterization

Now we will obtain a characterization of the class of Hausdorff convergence spaces having the largest Hausdorff compactifications

Theorem 2 8 A Hausdorff convergence space S has the largest Hausdorff compactification if and only if S has at most finitely many non-convergent ultrafilters

Proof We first prove the necessary part The proof follows in the following steps (i) Let (X,f) be the largest Hausdorff compactification of S Then, from Theorem 2 7, we know that S is locally compact (ii) Let (X_1,i) be the Richardson compactification of S Then $(X,f) \geq (X_1,i)$ Let h be the continuous function from X onto X_1, such that $h \circ f = i$ Let us define a map g from X_1 to X as follows

$g(x) = f(x)$, for x in S

$g(v) = \lim f(v)$, for a non-convergent ultrafilter v on S

Here lim stands for the unique limit to which the ultrafilter f(v) on the compact set X will converge

$$\begin{tikzcd} X \arrow[rr, "h"] & & X_1 \arrow[ll, bend right, "g"'] \\ & S \arrow[ul, "f"] \arrow[ur, "i"'] & \end{tikzcd}$$

This function g is well defined We will show that g is onto X and is one to one Let y belongs to X, since $f(S)$ is dense in X, there exists an ultrafilter ϕ on X containing $f(S)$ which X-converges to y Now $f^{-1}(\phi \cap f(S))$ is a non-convergent ultrafilter on S, and hence, is an element of $X_1 - i(S)$ Therefore, $g(f^{-1}(\phi \cap f(S))) = \lim f(f^{-1}(\phi \cap f(S))) = \lim \phi = y$ Hence, g is onto X Now, $h(g(x)) = h(f(x)) = i(x) = x$, and $h(g(v)) = h(\lim f(v)) = \lim (h(f(v)) = \lim i(v) = v$ Hence g is one to one Now, this function g enables us to identify X and X_1 as sets Hence, we can consider (X,f) to be consisting of the set X_1 and f to be defined as $f(x) = x$ from S into X

(iii) Let us define a new convergence structure on X (and denote the new space by X') as follows

let ϕ be a filter on X, then

ϕ X'-converges to x if and only if ϕ X-converges to x

ϕ X'-converges to v if and only if $\phi \geq v \wedge \psi$, where ψ is an ultrafilter on X which X-converges to v

We obviously see that (X',f) is a Hausdorff compactification of S and that $X' \geq X$, implying thereby that $(X',f) \geq (X,f)$ But by assumption $(X,f) \geq (X',f)$ Hence, by Theorem 1 21, $(X,f) = (X',f)$

and thereby $X = X'$ From this we see that a filter ϕ on X will X-converge to v if and only if $\phi \geq v \wedge \psi$, where ψ is an ultrafilter on X which X-converges to v (iv) Let v be a non-convergent ultrafilter on S, then $f(v)$ is an ultrafilter on X, and hence must X-converge But $f(v)$ cannot converge to a point in $f(S)$, let it converge to some v' in $X-f(S)$ Then $h \circ f(v)$ will X_1-converge to $h(v')$ That is, $i(v)$ X_1-converges to $h(v')$ But $i(v)$ X_1-converges to v and X_1 is a Hausdorff convergence space, and therefore $h(v') = v$ Hence, $g \circ h(v') = g(v) = v$ But $g \circ h =$ Identity on X , $v' = v$ Now, $f(v)$ X-converging to v, implies from (iii), that $f(v) \geq v \wedge \psi$, where ψ is an ultrafilter which X-converges to v Since $f(v) \neq v$, we have $f(v) = \psi$ Thus, we see that a filter ϕ on X will X-converge to v if and only if $\phi \geq v \wedge f(v)$ (v) Let ϕ be an ultrafilter on X containing $X-f(S)$ ϕ must X-converge, but cannot converge to a point in $f(S)$ because S is locally compact as mentioned in step (i) above, and hence $f(S)$ is X-open and ϕ contains $X-f(S)$ Let ϕ X-converge to some v in $X-f(S)$ Then as shown in step (iv) above, $\phi \geq v \wedge f(v)$ Since $\phi \neq f(v)$, we conclude that $\phi = v$ That is each ultrafilter on $X-f(S)$ is a point filter and hence by Theorem 1 2, $X-f(S)$ is finite, that is, S has at most finitely many non-convergent ultrafilters

Next, to prove the sufficiency part, let us assume that S has only finitely many non-convergent ultrafilters Let us denote by Y, the set of all x for x in S and all non-convergent ultrafilters on S

Let k be a function from S to Y defined as $k(x) = x$ Let us define a convergence structure on Y(and denote the resulting space by Y) as follows

let ϕ be a filter on Y, then

ϕ Y-converges to x if and only if $k(S)$ belongs to ϕ and $k^{-1}(\phi \cap k(S))$ S-converges to x in S

ϕ Y-converges to v if and only if $\phi \geq v \wedge k(v)$, for non-convergent ultrafilters v on S

It is easy to verify that Y is a convergence space To see that it is Hausdorff, let ϕ be an ultrafilter on Y which is simultaneously Y-converging to (a) x_1 and x_2 Then $k^{-1}(\phi \cap k(S))$ S-converging to x_1 and x_2, but S is Hausdorff, hence $x_1 = x_2$, (b) x and v, where v is a non-convergent ultrafilter on S Then $k(S)$ belongs to ϕ and $k^{-1}(\phi \cap k(S))$ S-converges to x, and $\phi \geq v \wedge k(v)$ This implies $\phi \neq v$, hence $\phi = k(v)$ But then, $k^{-1}(\phi \cap k(S)) = v$, which is a non-convergent ultrafilter on S, hence a contradiction, (c) v_1 and v_2, where v_1 and v_2 are two non-convergent ultrafilters on S Then $\phi \geq v_1 \wedge k(v_1)$ and $\phi \geq v_2 \wedge k(v_2)$ If $v_1 \neq v_2$, then $\phi \neq v_1$ and $\phi \neq v_2$, this implies that $\phi = k(v_1)$ and $\phi = k(v_2)$ But then, by definition of k, we see $v_1 = v_2$, a contradiction To see that Y is compact, let ϕ be an ultrafilter on Y If $Y\text{-}k(S)$ belongs to ϕ, then since $Y\text{-}k(S)$ is finite, by Theorem 1 2, ϕ is a point filter for some point in $Y\text{-}k(S)$, and hence convergent If $k(S)$ belongs to ϕ , then (a) if

$k^{-1}(\phi \cap k(S))$ S-converges to some x in S, then by definition ϕ Y-converges to x, (b) if $k^{-1}(\phi \cap k(S))$ is non-convergent on S, then this being an ultrafilter, ϕ will Y-converge to $k^{-1}(\phi \cap k(S))$ by definition Obviously the function k is a bicontinuous one to one function from S into Y $k(S)$ is dense in Y, because for v in $Y-k(S)$, $k(v)$ contains $k(S)$ and Y-converges to v by definition Therefore, we find that (Y,k) is a Hausdorff compactification of S Now, we will show that (Y,k) is the largest Hausdorff compactification of S To show this we will prove that if Z is a compact Hausdorff convergence space and s is a continuous function from S into Z, then this function s can be extended to a unique continuous function from Y onto the closure of $s(S)$ in Z For this, let us define a function t from Y to Z as follows

$t(x) = s(x)$, for x in S

$t(v) = \lim s(v)$, for v in $Y-k(S)$

$$\begin{tikzcd} Y \arrow[rr, dashed, "t"] & & Z \\ & S \arrow[ul, "k"] \arrow[ur, "s"'] & \end{tikzcd}$$

Now, we can verify that t is a well defined function onto the closure of $s(S)$ in Z such that $t \circ k = s$ To see that t is continuous, let ϕ be a filter on Y which Y-converges to x Then, by definition $k(S)$ belongs to ϕ and $k^{-1}(\phi \cap k(S))$ S-converges to x Hence, $s \circ k^{-1}(\phi \cap k(S))$ Z-converges to $s(x)$, and hence, $t(\phi)$ Z-converges to $t(x)$ Let ϕ Y-converges to v for some non-convergent

ultrafilter v on S, then $\phi \geq v \wedge k(v)$, and therefore, $t(\phi) \geq t(v \wedge k(v))$ $= t(v) \wedge t \circ k(v) = t(v) \wedge s(v)$, which Z-converges to $t(v)$, and hence, $t(\phi)$ Z-converges to $t(v)$ Now, uniqueness of the function t follows from Theorem 1 20 This completes the proof of the theorem

CHAPTER III

ON THE RICHARDSON COMPACTIFICATION

Introduction

Richardson constructed a compactification of a convergence space in [14] This was out lined in Chapter One This is a Hausdorff compactification, showing also that every Hausdorff convergence space has a Hausdorff compactification This compactification has the following property let S be a Hausdorff convergence space and (X_1, i) be its Richardson compactification Let f be a continuous function from S into a regular compact convergence space Y Then, there exists a continuous function h from X_1 onto the closure of $f(S)$ in Y such that $h \circ i = f$

$$\begin{tikzcd} X_1 \arrow[rr, "h"] & & Y \\ & S \arrow[ul, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Now, we note the following two things (a) (X_1, i) is, in general, not regular, and therefore, fails to be the largest regular compactification of S, (b) the regularity requirement on the convergence space Y cannot be relaxed, and therefore, (X_1, i)

fails to be the largest Hausdorff compactification of S

These observations make us study Hausdorff as well as regular compactifications of convergence spaces with Richardson compactification appearing in both In the domain of regular compactifications, the problem is under what additional conditions, will (X_1, i) become regular, and thereby, the largest regular compactification This problem has been solved by Gazik in [5] , where he has obtained a necessary and sufficient condition for (X_1, i) to be regular This condition is in terms of some requirement on non-convergent ultrafilters on the underlying convergence space He has further shown that his condition is also both necessary and sufficient for the Richardson compactification of a Tychonoff topological space may become topological, and thereby, coincide with the Stone-Čech compactification In section one of this chapter we mention Gazik's results without proof Next, in the domain of Hausdorff compactifications, the problem is, to characterize the class of Hausdorff convergence spaces for which the Richardson compactification becomes the largest Hausdorff compactification In section two we obtain such a characterization This characterization is in terms of local compactness as defined in the previous chapter and some condition on non-convergent ultrafilters on the underlying convergence space

1 *As The Largest Regular Compactification*

Let S be a regular convergence space Let us denote by R(S) its Richardson compactification Then Gazik [5] has proved the following result

<u>Theorem</u> 3 1 R(S) is regular if and only if $v = Cl(v)$, for each non-convergent ultrafilter v on S

Here $Cl(v)$ denotes the filter whose base is $Cl(V)$ for V in v, where closure has been taken in S

If S is a principal convergence space, then, by construction, R(S) is also principal Therefore, if R(S) is regular, then by Theorem 1 23, Theorem 1 17 and Theorem 1 24, R(S) is a compact Hausdorff topological space, and hence, S is a Tychonoff topological space If S is a Tychonoff topological space, and $v = Cl(v)$ for every non-convergent ultrafilter v on S, then as argued above, R(S) is a topological compactification of S And hence, by virtue of its property in regard to the extension of continuous functions, we see that $R(S) = \beta(S)$, where $\beta(S)$ is the Stone-Čech compactification of S On the other hand, if S is a Tychonoff topological space, and $R(S) = \beta(S)$, then R(S) is regular, and hence, $v = Cl(v)$ for every non-convergent ultrafilter v on S, by Theorem 3 1 above The following result is due to Gazik [5]

<u>Theorem</u> 3 2 If S is a Tychonoff topological space, then $R(S) = \beta(S)$ if and only if $v = Cl(v)$ for every non-convergent ultrafilter v on S

2 *As The Largest Hausdorff Compactification*

In this section we will obtain a necessary and sufficient condition that the Richardson compactification be the largest Hausdorff compactification of the underlying convergence space

<u>Characterization</u>

<u>Theorem</u> 3 3 If S is a Hausdorff convergence space, then the Richardson compactification (X_1, i) of S is the largest Hausdorff compactification of S if and only if the following two conditions are satisfied

(a) S is locally compact,

(b) $\overset{*}{v} = v \wedge i(v)$, for each non-convergent ultrafilter v on S

<u>Proof.</u> Let (X_1, i) be the largest Hausdorff compactification of S Then, by Theorem 2 7, S is locally compact Let us define a new convergence structure on X_1 (and denote the resulting new space by X_2) as follows

let ϕ be a filter on X_1, then

ϕ X_2-convergences to x if and only if ϕ X_1 - converges to x,

ϕ X_2-converges to v if and only if $\phi \geq v \wedge \psi$, where ψ is an ultrafilter on X_1 which X_1-converges to v

It is easy to see that (X_2, i) is a Hausdorff compactification of S and $(X_2, i) \geq (X_1, i)$ But by assumption $(X_1, i) \geq (X_2, i)$, hence by Theorem 1 21, $(X_1, i) = (X_2, i)$ and therefore, $X_1 = X_2$ This implies that a filter ϕ on X_1 will X_1-converge to v in $X_1 - i(S)$ if and only if $\phi \geq v \wedge \psi$, where ψ is an ultrafilter which X_1-converges to v Again, we know $i(v) \geq \hat{v}$, and hence $i(v) \geq v \wedge \psi$ But $i(v)$ is an ultrafilter on X_1 and $i(v) \neq v$, hence $i(v) = \psi$ Since $\hat{v}$ X_1-converges to v, we conclude that $\hat{v} \geq v \wedge i(v)$ But on the other hand $v \geq \hat{v}$ and therefore, $v \wedge i(v) \geq \hat{v}$ Thus we have shown that $\hat{v} = v \wedge i(v)$

Next, to prove that the two conditions are also sufficient, we will show that under these conditions every condinuous function k from S into a compact Hausdorff convergence space Y has a unique continuous extension f from X_1 onto the closure of $k(S)$ in Y, such that $f \circ i = k$

$$\begin{tikzcd} X_1 \arrow[rr, "f"] & & Y \\ & S \arrow[ul, "i"] \arrow[ur, "k"'] & \end{tikzcd}$$

Let us define a function f from X_1 to Y as follows

$f(x) = k(x)$, for x in S

$f(v) = \lim k(v)$, for non-convergent ultrafilter v on S

Now, it is easy to see that f is a well defined function onto the closure of $k(S)$ in Y such that $f \circ i = k$ We will show that f is continuous Let ϕ be a filter on X_1 which X_1- converges to x Then, condition (a) implies that $i(S)$ belongs to ϕ, and therefore, $i^{-1}(\phi \cap i(S))$ S-converges to x Hence, $k \circ i^{-1}(\phi \cap i(S))$ Y-converges to $k(x)$ That is $f(\phi)$ Y-converges to $f(x)$ Let ϕ X_1-converge to v Then $\phi \geq \hat{v}$, and condition (b) implies that $\phi \geq v \wedge i(v)$ Therefore, $f(\phi) \geq f(v \wedge i(v)) = f(v) \wedge f \circ i(v) = f(v) \wedge k(v)$, which implies $f(\phi)$ Y-converges to $f(v)$ Therefore, f is continuous, and since, uniqueness follows from Theorem 1 20, this completes the proof of the theorem

Remark

If a Hausdorff convergence space S has the largest Hausdorff compactification $L(S)$, then by Theorem 2 8, S has

only finitely many non-convergent ultrafilters Then the Hausdorff compactification (Y,k) of S constructed in the proof of the Theorem 2 8, is the largest Hausdorff compactification of S, and hence, $L(S) = (Y,k)$ Now, since (Y,k) has the property that every continuous function from S into a compact Hausdorff space has a unique extension to Y, so does $L(S)$ On the other hand, if some Hausdorff compactification of S has this property of extension of continuous functions, then it is obviousely the largest Hausdorff compactification of S Hence, the largest Hausdorff compactification is characterized by this property Therefore, whenever the Richardson compactification is the largest Hausdorff compactification of S, this compactification is characterized by the property of extension of continuous functions

CHAPTER IV

ON REGULAR COMPACTIFICATIONS

Introduction

In this chapter we study regular compactifications of convergence spaces In the last chapter we have seen a condition under which the Richardson compactificationof convergence space becomes regular But the more pertinent question is to characterize the class of convergence spaces which possess regular compactifications Such a characterization has been obtained by Richardson and Kent [15] Their condition is in terms of pretopological modification of the convergence space They have shown that if a convergence space has a regular compactification, then its pretopological modification is a Tychonoff topological space, and corresponding to each Hausdorff topological compactification of this Tychonoff space, we can construct one regular compactification of the convergence space They have further shown that the regular compactification corresponding to Stone-Cech compactification of the Tychonoff space is in fact the largest regular compactification Hence, each convergence space having regular compactifications, also possesses the largest regular compactification In section one of this chapter we mention the results of Richardson and Kent

without proof. In section two we investigate the problem of the smallest regular compactification. We prove that a convergence space possessing regular compactifications has the smallest regular compactification if and only if its pretopological modification is a locally compact topological space. We also define an equivalence relation on the class of all regular compactifications of a convergence space, and show that each equivalence class has a largest member and a smallest member. Each of this equivalence class, in fact, corresponds to a Hausdorff topological compactification of the pretopological modification of the convergence space, and the largest member of this equivalence class is the one obtained by Richardson and Kent.

1 *Largest Regular Compactification*

Richardson and Kent [15] have shown that the closure operator of a compact regular convergence space S is idempotent (Theorem 1.23), and hence, by Theorem 1.17, the pretopological modification of S is, in fact, the topological modification of S. Further they have shown that in this case the pretopological modification of S is also Hausdorff (Theorem 1.24). This leads to the following result.

<u>Theorem</u> 4.1 If (T,f) is a regular compactification of S, then $(\pi T, f)$ is a Hausdorff topological compactification of πS.

The main results of Richardson and Kent are the following

Theorem 4 2 A regular convergence space has regular compactification if and only if πS is a completely regular topological space, and each ultrafilter which is finer than the neighbourhood filter at x S-converges to x for all x in S

Theorem 4 3 If a convergence space has a regular compactification, then it has a Stone-Čech regular compactification

2 *Smallest Regular Compactification*

In this section we will obtain a characterization of the class of convergence spaces which possess smallest regular compactifications

Relation between the compactifications of S and πS

We call a regular convergence space having regular compactifications is R-convergence space In this section S will always denote an R-convergence space If (P,g) is a regular compactification of S, then as mentioned in Theorem 4 1, $(\pi P, g)$ is a Hausdorff topological compactification of πS Let K denote the set of all regular convergence structures γ on P which satisfy the following two conditions

(a) γ coincides with πP relative to ultrafilter convergence,

(b) if $g(S)$ belongs to a filter ϕ on P, then ϕ γ-converges to y in $g(S)$ if and only if $g^{-1}(\phi \cap g(S))$ S-converges to $g^{-1}(y)$

These two conditions are consistent, because, S being an R-convergence space, every ultrafilter finer than the neighbourhood filter at x S-converges to x for all x in S Let us denote by P_0 and P_1 the convergence spaces consisting of the set P equipped with the inf and sup of convergence structures in K respectively Now, it is easy to see that (P_0,g) and (P_1,g) are regular compactifications of S Further, $\pi P_0 = \pi P_1 = \pi P$, and $P_1 \geq P \geq P_0$ On the other hand, if P' is a regular convergence space such that $P_1 \geq P' \geq P_0$, then $\pi P_1 = \pi P' = \pi P_0$, and (P',g) is a regular compactification of S For the rest of the section (P_0,g) and (P_1,g) will stand for the regular compactifications of S as obtained here for a given regular compactification (P,g) of S

On (T_1, f)

If (T,f) is a Hausdorff topological compactification of the Tychonoff space πS, then Richardson and Kent [15] have

constructed a regular compactification (T_1, f) of S. By construction $T = \pi T_1$ and T_1 is the finest regular convergence space on the set T satisfying the two conditions (a) and (b) above for γ and πP replaced by T_1 and T respectively.

Now, we prove the following result.

<u>Theorem</u> 4.4. If (P, g) is a regular compactification of S and $(T, f) \geq (\pi P, g)$ as Hausdorff topological compactifications of πS, then $(T_1, f) \geq (P, g)$ as regular compactifications of S.

<u>Proof</u>. Let h be the continuous function from T onto πP such that $h \circ f = g$.

$$
\begin{tikzcd}
T \arrow[rr, dashed, "h"] & & \pi P \\
& \pi S \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T_1 \arrow[rr, dashed, "h"] & & P \\
& S \arrow[ul, "f"] \arrow[ur, dashed, "g"'] &
\end{tikzcd}
$$

We will show that h is a continuous function from T_1 onto P. If not, there exists a filter ϕ_1 on T such that ϕ_1 T_1-converges to some point x_1 in T, but $h(\phi_1)$ does not P-converge to $h(x_1)$. We define a new convergence structure on the set T (and denote the resulting space by T_1') as follows:

let ϕ be a filter on T, then

$\phi T_1'$-converges to $y \neq x_1$ in T if and only if

ϕT_1-converges to y,

$\phi T_1'$-converge to x_1 if and only if ϕ T_1-converges

to x_1 and $h(\phi)$ P-converges to $h(x_1)$

Now, $T_1' > T_1$ implies that T_1' is Hausdorff To see that T_1' coincides with T_1 (and hence with T) relative to ultrafilter convergence, we proceed as follows Since $T_1' > T_1$, we need only prove that if an ultrafilter ϕ on T is T_1-convergent to y, then ϕ is T_1'-convergent to y If $y \neq x_1$, by definition of T_1', this is true If $y = x_1$, then ϕT_1-converges to x_1 implies ϕ T-converges to x_1, because T_1 and T coincide relative to ultrafilter convergence Since h is a continuous function from T to πP, $h(\phi)$ πP-converges to $h(x_1)$ Now $P \geq \pi P$, and both P and πP are compact, Hausdorff spaces, and therefore P and πP coincide relative to ultrafilter convergence Since $h(\phi)$ is an ultrafilter on P, we see that $h(\phi)$ P-converges to $h(x_1)$ Therefore, by definition $\phi T_1'$ - converges to x_1 This condition further implies that the closure operators of T_1', T_1 and T are same We show that T_1' is regular as follows (i) if a filter ϕ on T is T_1'-convergent to $y \neq x_1$, then ϕ is T_1-convergent to y, since T_1 is regular, $Cl_{T_1} \phi$ T_1-converges to y But T_1-closure

is same as T_1'-closure, hence $Cl_{T_1'}\phi$ T_1'-converges to y by definition, (ii) if ϕ T_1'-converges to x_1, then ϕ T_1-converges to x_1, and since T_1 is regular, $Cl_{T_1}\phi$ T_1-converges to x_1, hence $Cl_{T_1'}\phi$ T_1-converges to x_1. Also $h(\phi)$ P-converges to $h(x_1)$, implies $Cl_P\, h(\phi)$ P-converges to $h(x_1)$. As pointed out above P and πP coincide relative to ultrafilter convergence, and hence, have same closure operators. Therefore, $Cl_{\pi P}\, h(\phi)$ P-converges to $h(x_1)$. Since h is continous from T to πP, Theorem 1.14 gives that $h(Cl_T\phi) \geq Cl_{\pi P}\, h(\phi)$. Therefore, $h(Cl_T\phi)$ P-converges to $h(x_1)$. That is, $h(Cl_{T_1'}\phi)$ P-converges to $h(x_1)$. Hence, by definition $Cl_{T_1'}\phi$ T_1'-converges to x_1. That is, T_1' is regular. Next, let $f(S)$ belongs to a filter ϕ on T, then ϕ T_1'-converges to some y in $f(S)$ if and only if $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$. To see this, we note that if $f(S)$ belongs to ϕ and ϕ T_1'-converges to y in $f(S)$, then ϕ T_1-converges to y in $f(S)$, and by virtue of property (a) of T_1, we know that $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$. On the other hand if $f^{-1}(\phi \cap f(S))$ S-converges $f^{-1}(y)$, then again by the same property, ϕ T_1-converges to y. And also $g \circ f^{-1}(\phi \cap f(S))$ P-converges to $g \circ f^{-1}(y)$, because (P, g) is a compactification of S. That is $h(\phi)$ P-converges to $h(y)$. Hence, ϕ T_1'-converges to y. Therefore, we see that T_1' is a regular convergence space on the set T satisfying the two conditions (a) and (b) for γ and πP replaced by T_1' and T.

respectively Also $T_1' > T$, but this is a contradiction of the fact that T_1 is the finest such convergence structure on the set T Hence, h is a continuous function from T_1 onto P such that $h \circ f = g$ That is, $(T_1,f) \geq (P,g)$ as regular compactifications of S This completes the proof of the theorem

The result of Richardson and Kent

If (T,f) is the Stone-Čech compactification of the Tychonoff space πS, and if (P,g) is a regular compactification of S, then $(T,f) \geq (\pi P,g)$ as topological compactifications of πS Hence, by Theorem 4 4 $(T_1,f) \geq (P,g)$ as regular compactifications of S This implies that (T_1,f) is the largest regular compactification of S Hence, each R-convergence space has a Stone-Cech regular compactification

On (T_0,f)

Let us denote by T_0 the convergence space consisting of the set T equipped with the inf of all regular convergence structures γ on T satisfying the two conditions (a) and (b) above for πP replaced by T Then as pointed out above (T_0,f) is a regular compactification of S such that $\pi T_1 = \pi T_0 = T$ Below we give an alternative description of T_0

Theorem 4 5 T_o is the finest convergence space on the set T having the following properties

(i) if y does not belong to $f(S)$, then $V_T(y)$ T_o-converges to y,

(ii) if y belongs to $f(S)$ and trace of $V_T(y)$ on $T-f(S)$ exists, then $Cl_T\, \phi \wedge [V_T(y) \cap (T-f(S))]$ T_o-converges to y, where $f(S)$ belongs to the filter ϕ on T and $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$

(iii) if y belongs to $f(S)$ and trace of $V_T(y)$ on $T-f(S)$ does not exist, then $Cl_T\, \phi$ T_o-converges to y, where $f(S)$ belongs to the filter ϕ on T and $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$

Proof The proof follows in two steps, firstly we will show that T_o satisfies the conditions (i), (ii) and (iii) of the theorem, and secondly that T_o is the finest such convergence space We see that condition (iii) is satisfied by T_o by construction Suppose T_o does not satisfy the condition (i), that is, there exists a point y_o in $T-f(S)$ such that $V_T(y_o)$ does not T_o-converge to y_o Then let us define a new convergence structure on T(and denote it by T_o') as follows

let ϕ be a filter on T, then

$\phi T_o'$-converges to $x \neq y_o$ if and only if ϕ T_o-converges to x,

$\phi T_o'$-converges to y_o if and only if $\phi \geq V_T(y_o)$

Then obviously $T_o > T'_o$ We see that if ϕ is an ultrafilter T'_o-converging to y_o, then $\phi \geq V_T(y_o)$, and therefore, ϕ T-converges to y_o Hence, we can verify that T'_o coincide with T_o (and therefore with T) relative to ultrafilter convergence This implies that T'_o is Hausdorff and closure operators of T'_o, T_o and T are the same Now, let a filter ϕ T'_o-converges to y_o implies $\geq V_T(y_o)$, and therefore ϕ T-converges to y_o, and therefore, $Cl_T\,\phi$ T-converges to y_o, that implies $Cl_T\,\phi \geq V_T(y_o)$, therefore $Cl_{T'_o}\,\phi \geq V_T(y_o)$, and hence, $Cl_{T'_o}\,\phi$ T'_o-converges to y_o This shows that T'_o is regular Since y_o does not belong to $f(S)$, we see that T'_o satisfies the conditions (a) and (b) above for Υ and πP replaced by T'_o and T respectively Since $T_o > T'_o$, and T_o is the coarsest convergence space satisfying these conditions, we get a contradiction Hence, T_o satisfies condition (i) of the theorem Next, suppose T_o does not satisfy the condition (ii) Then there exists a point y_o in $f(S)$ and a filter ϕ_o on T such that $f(S)$ belongs to ϕ_o and $f^{-1}(\phi_o || f(S))$ S-converges to $f^{-1}(y_o)$, and $V_T(y_o) \sqcap (T-f(S))$ exists, but $Cl_T\,\phi_o \wedge [V_T(y_o) \sqcap (T-f(S))]$ does not T_o-converge to y_o Let us define a new convergence structure on the set T (and denote it by T'_o) as follows

let ϕ be a filter on T, then

$\phi T'_o$-converges to $y \neq y_o$ if and only if ϕ

T_o- converges to y,

$\phi\ T_o'$-converges to y_o if and only if either $\phi\ T_o$-converges to y_o or $\phi \geq Cl_T\ \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$

It is simple to see that T_o' is a convergence space, for this we need only note that ϕ_o T-converges to y_o and therefore $y_o \geq Cl_T\ \phi_o$. By construction $T_o > T_o'$. To prove that T_o'-coincides with T_o (and therefore with T) relative to ultrafilter convergence, we need only note that $[V_T(y_o) \cap (T-f(S))] \geq V_T(y_o)$, and hence $[V_T(y_o) \cap (T-f(S))]$ T-converges to y_o. This implies that T_o' is Hausdorff and closure operators of T_o', T_o and T are same. This in turn implies that T_o' is regular, because $Cl_T\ \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$ is a T-closed filter on T. To see that T_o' satisfies condition (b) above for γ and πP replaced by T_o' and T respectively, we need only show that if $f(S)$ belongs to ϕ and $\phi \geq Cl_T\ \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$, then $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y_o)$. Let ϕ' be an ultrafilter on T such that $\phi' \geq \phi$. Since $f(S)$ belongs to ϕ', we see that $\phi' \not\geq [V_T(y_o) \cap (T-f(S))]$. Hence $\phi' \geq Cl_T\ \phi_o$. This implies $\phi \geq Cl_T\ \phi_o$. Now, $f^{-1}(\phi_o \cap f(S))$ S-converges to $f^{-1}(y_o)$ implies ϕ_o T_o-converges to y_o, and T_o is regular, therefore $Cl_T\ \phi_o$ T_o-converges to y_o. Hence, ϕ T_o'-converges to y_o. Thus T_o' satisfies the conditions (a) and (b) above for γ and πP replaced by T_o' and T respectively and also $T_o > T_o'$. This gives a contradiction because T_o is the coarsest such space. Therefore, T_o satisfies

condition (ii) of the theorem Next, we will show that if some convergence space T_o' on T satisfies conditions (i), (ii) and (iii) of the theorem, then $T_o \geq T_o'$ Let a filter ϕ on T is T_o-convergent to y not in $f(S)$ Then T being pretopological modification of T_o, ϕ T-converges to y, and therefore $\phi \geq V_T(y)$ Hence, by condition (i) ϕ T_o'-converges to y Let ϕ T_o-converges to y in $f(S)$, and let (A_1) $f(S)$ belongs to ϕ and $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$, then ϕ T_o'-converges to y by condition (ii) or (iii) according as $V_T(y) \cap (T-f(S))$ exists or not; (A_2) if $T-f(S)$ belongs to ϕ, then $V_T(y) \cap (T-f(S))$ exists and $\phi \geq [V_T(y) \cap (T-f(S))]$ Hence, by condition (ii) ϕ T_o'-converges to y, (A_3) if neither $f(S)$ nor $T-f(S)$ belongs to ϕ, then $V_T(y) \cap (T-f(S))$ does exist Let $\phi_1 = [\phi \cap f(S)]$ and $\phi_2 = [\phi \cap (T-f(S))]$ Then, $\phi = \phi_1 \wedge \phi_2$ Now, $f(S)$ belongs to $\phi_1 \geq \phi$, therefore ϕ_1 T_o-converges to y and hence $f^{-1}(\phi_1 \cap f(S))$ S-converges to $f^{-1}(y)$ And $T-f(S)$ belongs to $\phi_2 \geq \phi \geq V_T(y)$, therefore, $\phi_2 \geq [V_T(y) \cap (T-f(S))]$ Hence, by condition (ii) ϕ T_o'-converges to y Thus we conclude that T_o is the finest convergence space on T satisfying (i), (ii) and (iii) This completes the proof of the theorem

Now using Theorem 4 5, we will prove a theorem for (T_o,f) which is analogous to Theorem 4 4 for (T_1,f)

Theorem 4.6 If (P,g) is a regular compactification of S and $(\pi P,g) \geq (T,f)$ as Hausdorff topological compactifications of πS, then $(P,g) \geq (T_0,f)$ as regular compactifications of S.

Proof. Let h be the continuous function from πP onto T such that $h \circ g = f$. We will show that h is continuous from P onto T_0.

$$\begin{tikzcd} \pi P \arrow[rr, dashed, "h"] & & T \\ & \pi S \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[rr, "h"] & & T_0 \\ & S \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

Let ϕ be a filter on P which P-converges to some y not in $g(S)$. Then ϕ πP-converges to y, and hence, $h(\phi)$ T-converges to $h(y)$ and this implies that $h(\phi) \geq V_T(h(y))$. Now by Theorem 1.22, $h(y)$ does not belong to $f(S)$, and therefore, by condition (i) of Theorem 4.5, $h(\phi)$ T_0-converges to $h(y)$. Now, let ϕ P-converge to y in $g(S)$. Then, (A_1) if $g(S)$ belongs to ϕ, then this implies $g^{-1}(\phi \cap g(S))$ S-converges to $g^{-1}(y)$. This implies, $f \circ g^{-1}(\phi \cap g(S)) = h(\phi)$ T_0-converges to $f \circ g^{-1}(y)$ $= h(y)$, (A_2) if $g(S)$ does not belong to ϕ, let $\phi_1 = [\phi \cap (P - g(S))]$ and if the trace of ϕ on $g(S)$ also exists, then $\phi_2 = [\phi \cap g(S)]$. Then, $\phi = \phi_1 \wedge \phi_2$. Since ϕ P-converges to y, $h(\phi) \geq V_T(h(y))$. By Theorem 1.22, since $g(S)$ does not belong to ϕ, $f(S)$ does not

belong to $h(\phi)$ Hence $V_T(h(y)) \cap (T-f(S))$ exists Now, $\phi_1 \geq \phi$ which P-converges to y implies $h(\phi_1)$ T-converges to $h(y)$ This implies, $h(\phi_1) \geq V_T(h(y))$ Since $P-g(S)$ belongs to ϕ_1, $T-f(S)$ belongs to $h(\phi_1)$ by Theorem 1 22 again Hence, $h(\phi_1) \geq [V_T(h(y)) \cap (T-f(S))]$ Also, $g(S)$ belongs to $\phi_2 \geq \phi$ Hence ϕ_2 P-converges to y, this implies $g^{-1}(\phi_2 \cap g(S))$ S-converges to $g^{-1}(y)$ This implies $f \circ g^{-1}(\phi_2 \cap g(S))$ T_0-converges to $f \circ g^{-1}(y)$ Hence, $h(\phi_2)$ T_0-converges to $h(y)$ Again, by Theorem 1 22, $f(S)$ belongs to $h(\phi_2)$ Hence, we see that $h(\phi) \geq h(\phi_1) \wedge h(\phi_2) \geq [V_T(h(y)) \cap (T-f(S))] \wedge Cl_T(h(\phi_2))$ Thus, by condition (ii) of Theorem 4 5, $h(\phi)$ T_0-converges to $h(y)$ Hence, h is continuous from P onto T_0 and $h \circ g = f$ Therefore, $(P,g) \geq (T_0,f)$ This completes the proof

Characterization

If πS is a locally compact Hausdorff topological space and (T,f) is the Alexandroff one point Hausdorff topological compactification of πS, then from Theorem 4 6, we find that, if (P,g) is a regular compactification of S, then $(\pi P,g) \geq (T,f)$, and therefore, $(P,g) \geq (T_0,f)$ That is, (T_0,f) is the smallest regular compactification of S On the other hand, if S has the smallest regular compactification say (P,g), then we will show that $(\pi P,g)$ is the smallest Hausdorff topological compactification of the Tychonoff space πS, and hence, πS is a locally compact

Hausdorff topological space Let (T,f) be an arbitrary Hausdorff topological compactification of πS Let (T_1,f) be the Richardson-Kent regular compactification of S, corresponding to (T,f) of πS Then, $(T_1,f) \geq (P,g)$ Let h be the continuous function from T_1 onto P such that $h \circ f = g$ Then, from Theorem 1 16, h is continuous from πT_1 onto πP But $\pi T_1 = T$, hence $(T,f) \geq (P,g)$

This leads us to our main result

<u>Theorem</u> 4 7 If S is an R-convergence space, then S has the smallest regular compactification if and only if πS is a locally compact Hausdorff topological space

<u>Equivalence classes</u>

We will define an equivalence relation on the set of all regular compactifications of an R-convergence space , as follows if (P,g) and (Q,f) are two regular compactifications of S, then $(P,g) \sim (Q,f)$ if and only if $(\pi P,g) = (\pi Q,f)$ as Hausdorff topological compactifications of the Tychonoff space πS This gives us the following result

<u>Theorem</u> 4.8 If S is an R-convergence space, then each equivalence class of regular compactifications of S, has a largest member and a smallest member

<u>Proof</u> Let $(P,g) \sim (Q,f)$ Since $P_1 \geq P \geq P_o$ and $Q_1 \geq Q \geq Q_o$, we need only show that $(P_1,g) = (Q_1,f)$ and $(P_o,g) = (Q_o,f)$ First of these follows from Theorem 4 4 and the second from Theorem 4 6

CHAPTER V

ON m-ULTRACOMPACTIFICATIONS

Introduction

In this chapter our aim is to introduce m-ultracompactification. This is a generalisation of the Richardson compactification. We use Richardson's technique to construct the m-ultracompactification. m-ultracompactness for T_1 topological spaces have been defined by J. van der Slot [16], for an infinite cardinal m. $\aleph_0$-ultracompactness is equivalent to compactness. Realcompactness and $\aleph_1$-ultracompactness for countably compact normal spaces are equivalent. This definition is suitable for extension to convergence spaces and works well with Richardson method of compactification. In the first section we find condition for a convergence space to have smallest Hausdorff m-ultracompactification. In section two we construct a Hausdorff m-ultracompactification in a manner analogous to Richardson compactification. We show that if S is the convergence space and ρS the Hausdorff m-ultracompactification and f is a continuous function from S into a Hausdorff regular m-compact space, then f has a unique continuous extension to ρS. Then we find conditions under which ρS becomes the largest Hausdorff (regular) m-ultracompactification of S. When S is a Tychonoff topological space, we do not know condition under

which ρS becomes topological It is mentioned here as an open problem Two more open problems are also mentioned

1 *Smallest Hausdorff m-ultracompactification*

In this section we characterize the class of Hausdorff convergence spaces having smallest m-ultracompactification We first note that compact spaces are m-ultracompact for each infinite cardinal m Hence each Hausdorff convergence space has a one point Hausdorff m-ultracompactification (the one point Hausdorff compactification constructed in Chapter One will serve the purpose)

Local m-ultracompactness

<u>Definition</u> 5 1 A Hausdorff convergence space S is locally m-ultracompact if and only if it is open in each of its Hausdorff m-ultracompactifications

Characterization

<u>Theorem</u> 5 2 A Hausdorff convergence space has smallest Hausdorff m-ultracompactification if and only if S is locally m-ultracompact

which ρS becomes topological It is mentioned here as an open problem Two more open problems are also mentioned

1 *Smallest Hausdorff m-ultracompactification*

In this section we characterize the class of Hausdorff convergence spaces having smallest m-ultracompactification We first note that compact spaces are m-ultracompact for each infinite cardinal m Hence each Hausdorff convergence space has a one point Hausdorff m-ultracompactification (the one point Hausdorff compactification constructed in Chapter One will serve the purpose)

Local m-ultracompactness

Definition 5 1 A Hausdorff convergence space S is locally m-ultracompact if and only if it is open in each of its Hausdorff m-ultracompactifications

Characterization

Theorem 5 2 A Hausdorff convergence space has smallest Hausdorff m-ultracompactification if and only if S is locally m-ultracompact

Proof Let a Hausdorff convergence space S have the smallest Hausdorff m-ultracompactification (T',j) and let (T, i) be the one point Hausdorff compactification of S Then, $(T,i) \geq (T',j)$ By Theorem 1 22, we conclude that (T',j) is a one point m-ultracompactification and hence S is open in T' Another application of Theorem 1 22 together with Theorem 1 15, enables us to conclude that S is open in each of its m-ultracompactifications

Next, assuming S to be locally m-ultracompact, we can show that the one point compactification (T, i) is the smallest m-ultracompactification of S The proof follows exactly in the same line as in the proof of Theorem 2 2

2 *m-ultracompactification*

Construction

Let S be a Hausdorff convergence space and ρS the set of all x for x in S and all non-convergent m-ultrafilters on S If $\mathcal{F}$ is a filter on S, then we define a filter $\hat{\mathcal{F}}$ on ρS as the one having basis $\{\hat{F} | F \in \mathcal{F}\}$, where $\hat{F} = \{H \in \rho S | F \in H\}$ We define a convergence structure on ρS as follows

let ϕ be a filter on S , then

ϕ ρS-converges to x for x in S if and only if $\phi \geq \hat{\mathcal{F}}$ for some filter $\mathcal{F}$ on S which S-converges to x

ϕ ρS-converges to v, for non-convergent m-ultrafilter v on S if and only if $\phi \geq \hat{v}$. We note that $\hat{F}_1 \cap \hat{F}_2 = (F_1 \cap F_2)^\wedge$, $\hat{F}_1 \cup \hat{F}_2 = (F_1 \cup F_2)^\wedge$, $x \geq \dot{x}$, $v \geq \hat{v}$, using these we can see that ρS is a Hausdorff convergence space. To show that ρS is m-ultracompact, let ϕ be an ultrafilter on ρS and $C_{\rho S}$ denote the set of all ρS-closed subsets and $\phi \cap C_{\rho S}$ has m-intersection property. Let $F = \{F \subset S \mid \hat{F} \in \phi\}$, then F is ultrafilter on S. We will show that F is an m-ultrafilter on S. Let $F_i \in F \cap C_S$, where $i \in I$ and $|I| < m$. We will first show that if A is S-closed then $\hat{A}$ is ρS-closed. Let ψ be a filter on ρS containing $\hat{A}$ which ρS-converges to some H in ρS. (i) If $H = x$, $\psi \geq \hat{F}$ for some F S-converging to x. This implies $\hat{A} \cap \hat{F} \neq \emptyset$ for every F in F. This implies $A \cap F \neq \emptyset$ for every F in F. Let G be a filter on S containing A and $G \geq F$, then G S-converges to x and hence x belongs to $Cl_S A = A$. Therefore, $H = x \in \hat{A}$. (ii) If $H = v$ for some v in $\rho S - S$, then $\psi \geq \hat{v}$. This implies $\hat{A} \cap \hat{V} \neq \emptyset$ for every V in v. Hence $A \cap V \neq \emptyset$ for every V in v, but v is an ultrafilter on S, therefore A belongs to v. Hence $H = v \in \hat{A}$. Now since F_i's are S-closed, $\hat{F}_i$'s are ρS-closed. And $\hat{F}_i \in \phi$ by definition of F. Hence $\hat{F}_i \in \phi \cap C_{\rho S}$. Now $\phi \cap C_{\rho S}$ has m-intersection property, therefore $\bigcap \hat{F}_i \neq \emptyset$. Let $H \in \bigcap \hat{F}_i$, then $H \in \hat{F}_i$ for each i, that is, $F_i \in H$ for each i. Now each $H \in \rho S$ is an m-ultrafilter on S, hence $\bigcap F_i \neq \emptyset$. Hence F has m-intersection property. By construction

$\phi \geq \hat{F}$ Now F on S either S-converges to some x in S or F is a non-convergent m-ultrafilter on S, in the first case ϕ ρS-converges to x and in the second case ϕ ρS-converges to $F \in \rho S - S$ Let us define a function i from S to ρS as $i(x) = x$ We note that $i(F) = \hat{F} \cap i(S)$ for filter F on S From this we can easily see that i is an isomorphism from S to ρS Further if $v \in \rho S - S$, then $i(v) \geq \hat{v}$ ρS-converges to v and contains $i(S)$, hence $i(S)$ is dense in ρS Finally let f be a continuous function from S into a regular m-ultracompact convergence space T', then let us define f_1 from ρS to T' as follows

$$f_1(x) = f(x)\text{, for } x \text{ in } S$$
$$f_1(v) = \lim f(v)\text{, for } v \text{ in } \rho S - S$$

We note that if $v \in \rho S - S$, then $v \cap C_S$ has m-intersection property and v is a non-convergent ultrafilter on S We will show that $f(v)$ is an m-ultrafilter on T', then $f(v)$ will T'-converge to a unique point, that point is defined as $f_1(v)$ Let $A_i \in f(v) \cap C_{T'}$, $i \in I$ and $|I| < m$ Then there exist F_i's in v such that $f(F_i) \subset A_i$ Therefore $f^{-1}f(F_i) \subset f^{-1}(A_i)$, that is, $F_i \subset f^{-1}f(F_i) \subset f^{-1}(A_i)$ Now A_i's being T'-closed and f continuous from S to T', $f^{-1}(A_i)$ are S-closed and also belongs to v This implies, $\cap f^{-1}(A_i) \neq \emptyset$, therefore $\cap A_i \neq \emptyset$ Hence $f(v)$ is an m-ultrafilter on T' To see that f_1 is continuous from ρS to T', let ϕ be a filter on ρS which is

ρS-convergent to x , then $\phi \geq \hat{F}$ for some filter F on S which S-converges to x Hence $f(F)$ T'-converges to $f(x)$ Since T' is regular $Cl_{T'}$ $f(F)$ T'-converges to $f(x)$ Now since $f(x) = f_1(x)$, it is sufficient to show that $f_1(\phi) \geq Cl_{T'}f(F)$ In case ϕ ρS-converges to v in ρS-S, $\phi \geq \hat{v}$ as shown above $f(v)$ is a m-ultrafilter on T', hence must converge, let it T'-converges to y, then $Cl_{T'}f(v)$ also T'-converges to y It is sufficient to show that $f_1(\phi) \geq Cl_{T'}f(v)$, because $f_1(v) = \lim f(v) = y$ Now, let ϕ be a filter on ρS such that $\phi \geq \hat{F}$ for some filter F on S, then we will show that $f_1(\phi) \geq Cl_{T'}f(F)$ Let F belongs to F , then there exists an F^* in ϕ such that $F^* \subset \hat{F}$ We will show that $f_1(F^*) \subset Cl_{T'}f(F)$ Let v belongs to F^*, then v belongs to $\hat{F}$, that is, F belongs to v, therefore $f(F)$ belongs to $f(v)$ Since $f_1(v) = \lim f(v)$, $f_1(v)$ belongs to $Cl_{T'}f(F)$ Hence f_1 is continuous from ρS to T' Obviously $f_1 \circ i = f$ Uniqueness of f_1 follows from Theorem 1 20

$(\rho S, i)$ as largest Hausdorff m-ultracompactification

Theorem 5 4 $(\rho S, i)$ is the largest Hausdorff m-ultracompactification of a Hausdorff convergence space S if and only if the following two conditions are satisfied

(a) S is open in ρS

(b) $\hat{v} = v \wedge i(v)$, for v in ρS-S

ρS-converges to v. Therefore, $Cl_{\rho S}\, i(v)$ ρS-converges to v. This implies $Cl_{\rho S} i(v) \geq \hat{v}$. Hence if V belongs to v, then there exists a V' in v such that $Cl_{\rho S} i(V') \subset \hat{V}$. We will show that $Cl_S V' \subset V$. Let x belong to $Cl_S V'$, there exists a filter $\mathcal{F}$ on S containing V' which S-converges to x. This implies, $i(\mathcal{F})$ ρS-converges to $i(x) = x$ and $i(V')$ belongs to $i(\mathcal{F})$. Hence x belongs to $Cl_{\rho S} i(V') \subset \hat{V}$. That says, x belongs to V. Therefore, we conclude that $Cl_S v \geq v$, hence $Cl_S v = v$ for each v in $\rho S - S$.

Open Problems

(1) Under what conditions will ρS become topological for Tychonoff space S.

(2) Characterization of convergence spaces having largest Hausdorff m-ultracompactifications.

(3) Characterization of convergence spaces having regular m-ultracompactifications. When will these spaces have largest (smallest) m-ultracompactifications.

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PUBLICATIONS

The following two papers have appeared in print containing some of the results of the present thesis

(1) On smallest compactification for convergence spaces,
Proc Amer Math Soc 44 (1974), 225 - 230
The first section of this paper is on smallest Hausdorff compactification
(Section One of Chapter Two)
The second section of this paper is on smallest regular compactification
(Section Two of Chapter Four)

(2) On largest Hausdorff compactification for convergence spaces
Bull Austral Math Soc 12 (1975), 73 - 79
The first section of this paper is on Richardson compactification
(Section Two of Chapter Three)
The second section of this paper is on largest Hausdorff compactification
(Section Two of Chapter Two)